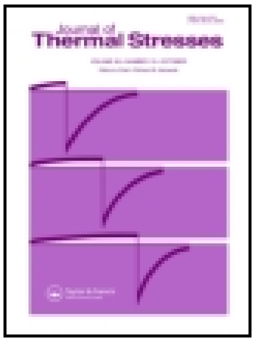

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Waves in magneto-thermoelastic solids under modified Green–Lindsay model

Nihar Sarkar[a] and Soumen De[b]

[a]Department of Mathematics, City College, Kolkata, India; [b]Department of Applied Mathematics, University of Calcutta, Kolkata, India

ABSTRACT
In this paper, the propagation of time-harmonic plane waves is investigated in an infinite elastic solid material by employing the modified Green–Lindsay (MGL) model of generalized thermoelasticity. It is found that three basic waves consisting of two sets of coupled longitudinal waves and one independent vertically shear-type (SV-type) wave may travel with distinct speeds. The sets of coupled waves are found to be dispersive, attenuating and influenced by the thermoelastic coupling effect. In contrast to the Green–Lindsay (GL) as well as the Lord–Shulman (LS) models, the SV-type wave is not only dispersive in nature but also experiences attenuation. Reflection phenomenon of an incident coupled longitudinal wave from stress-free and thermally insulated boundary surface of a thermoelastic solid half-space is addressed. Using these boundary conditions, the formulae for various reflection coefficients and their respective energy ratios are presented. For a particular model, various graphs are plotted to analyze the behavior of the phase speeds, reflection coefficients and their respective energy ratios. The characteristics of employing the MGL model are discussed by comparing the numerical results obtained for the present model with those obtained in the case of the GL model.

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Magneto-thermoelasticity; plane waves; dispersion; reflection; energy partition

Introduction

Over the last five decades, thermoelastic models in which thermal signals propagate at finite velocity have attracted much attention. The classical coupled thermoelasticity (CTE) model proposed by Biot [1] with the introduction of the strain-rate term in the classical Fourier's heat conduction law leads to a diffusion-type heat conduction model. In this model, although the elastic wave propagates with finite phase speed, the thermal wave propagates at an infinite phase speed, which is not possible physically. In order to solve this serious issue, in 1967, Lord and Shulman [2] proposed a generalized thermoelastic model (LS model) based on the Maxwell–Cattaneo [3] generalized model of Fourier's law of heat conduction. After the notable work of Lord and Shulman, Green and Lindsay [4] added temperature rate among the constitutive variables and developed another model (GL model), labeled as temperature-rate-dependent generalized thermoelasticity or generalized thermoelasticity with two relaxation times. Both of these theories predict the finite speed of propagation of the elastic as well as the thermal wave.

Recently, Yu et al. [5] established a new model of generalized thermoelasticity theoretically by considering the strain rate term in the Green–Lindsay (GL) model [4] of generalized

CONTACT Soumen De soumenisi@gmail.com Department of Applied Mathematics, University of Calcutta, Kolkata, 700009, India.
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thermoelasticity with the aid of extended thermodynamics. A problem of semi-infinite one-dimensional thermoelastic medium with traction free at one end and subjected to a temperature rise using the Laplace transform method is also studied by Yu et al. [5]. They observed that the strain rate may eliminate the discontinuity of the displacement at the elastic and thermal wave-fronts. They compared the present model [5] with the GreenNaghdi (GN) models [6,7] and conclude that the thermal wave speed of the present model is faster than the GN II model [7], and slower than the GN III model [6]. They also showed that their new model is free from the jump discontinuity that occurred in the displacement distribution in the case of GL model, and is safer in engineering practices than the GN model. Later, Quintanilla [8] reported some qualitative results for the modified Green–Lindsay (MGL) thermoelasticity [5]. He proved the exponential decay of the solutions and given a description of the spatial behavior of the solutions of the modified Green–Lindsay thermoelasticity. He also deduced a result of the continuous dependence of the solutions with respect to the initial conditions and to the supply terms and established the exponential stability of the solutions with respect to the time in the case where the supply terms vanish.

The study of propagation of seismic waves in thermoelastic media is of great importance in various fields such as earthquakes, geophysics, soil dynamics, seismology etc. Wave phenomenon in a thermoelastic medium is of great practical importance in various technological and geophysical circumstances. The propagation of waves along with other geophysical and geothermal data carries information about the structure and distribution of underground magnum. The wave propagation as part of exploration seismology helps in various economic activities like tracing of hydrocarbons and other mineral ores which are essential for various developmental activities like construction of dams, huge buildings, roads, bridges, the design of highways as well as foundation problems in soil mechanics. The problem of reflection of plane waves has been the subject of several investigations. Some of the notable works on waves in thermoelastic media are listed in the kinds of literature [9–18]. Othman and Song [19–22] discussed some reflection problems of thermoelastic waves under different conditions. The reflection of coupled generalized temperature rate-dependent thermoelastic waves on a half-space was investigated by Gupta et al. [23]. Sing [24] studied wave propagation in a Green–Naghdi thermoelastic solid with diffusion. Allam et al. [25] applied GL model on reflection of P and SV-waves from the free surface of thermoelastic diffusion solid under influence of the electromagnetic field and initial stress. Abd-Alla et al. [26] also reported the reflection of plane waves from electro-magneto-thermoelastic half-space with a dual-phase-lag model. Othman et al. [27] investigated the reflection of plane waves from a rotating magneto-thermoelastic medium with two-temperature and initial stress under three theories. Biswas and Sarkar [28] derived the solution of the steady oscillation equations in porous thermoelastic medium with dual-phase-lag model. They also studied the phase velocity, attenuation coefficient and penetration depth of time-harmonic plane waves in porous thermoelastic medium with dual-phase-lag. Li et al. [29] investigated the reflection and transmission of elastic waves at an interface with consideration of couple stress and thermal wave effects. Reflection of generalized magneto-thermoelastic waves with two temperatures under the influence of thermal shock and initial stress has been discussed by Abo-Dahab [30]. Recently, Sarkar and Tomar [31] reported plane waves in nonlocal thermoelastic solid with voids. Waves propagation in dual-phase-lag thermoelastic materials with voids based on Eringen's nonlocal elasticity has been reported by Mondal and Sarkar [32]. Das et al. [33] investigated the reflection of plane waves from the stress-free isothermal and insulated boundaries of a nonlocal thermoelastic solid.

During our literature review, we noticed that no thermoelastic plane wave reflection problem has been studied so far in the context of the new modified Green–Lindasy theory [5]. In the present investigation, we study the reflection phenomenon of magneto-thermoelastic plane harmonic waves from the thermally insulated and stress-free surface of a homogeneous, isotropic thermally conducting solid half-space by employing the MGL theory of generalized thermoelasticity with strain rate, proposed by Yu et al. [5]. The thermoelastic coupling effect creates two types of coupled longitudinal waves which are dispersive as well as exhibit attenuation. Different from the thermoelastic coupling effect, there also exists one independent vertically shear-type (SV-type) wave. In contrast to the GL [4] and LS [2] theories of generalized thermoelasticity, the SV-type wave is not only dispersive in nature but also experiences attenuation. Analytical expressions for the reflection coefficients and their respective energy ratios for reflected thermoelastic waves are determined when a coupled longitudinal wave is made incident on the free surface. The paper concludes with the numerical results on the phase speeds, reflection coefficients and their respective energy ratios for specific parameter choices. Various graphs have been plotted to analyze the behavior of these quantities. The characteristics of employing the MGL model are discussed by comparing the numerical results obtained for the present model with those obtained in the case of the GL model of generalized thermoelasticity.

Governing equations and formulation of the problem

A thermoelastic process is a coupled dynamical process of an exchange of mechanical energy into thermal energy and vice-versa under the action of externally applied thermo-mechanical loading [34]. Such a process is accompanied by strain and temperature changes inside the body all of which vanish upon the removal of the applied loading. The process can be described in terms of the physical field variables like temperature, displacement vector and strain tensor.

We shall consider a linear, homogeneous, isotropic, thermally and electrically conducting thermoelastic half-space, namely $\Omega = \{(x, y, z); -\infty < x, y < \infty,\ 0 \le z < \infty\}$ at uniform reference temperature T_0 under the action of a uniform magnetic field of intensity $\vec{H}_0$ acting in the positive direction of y–axis, so that $\vec{H}_0 = (0, H_0, 0)$, where H_0 is a constant. Let the origin O of a fixed rectangular Cartesian coordinate system $Oxyz$ be fixed at a point on the plane boundary $z = 0$ with z–axis pointing vertically downward into Ω and x–axis is directed along the horizontal direction (see Figure 1). The y–axis is taken in the direction of the line of intersection of the plane wave front with the plane surface. The boundary surface $z = 0$ is assumed to be thermally insulated and free from mechanical stresses. Due to the application of magnetic field $\vec{H}_0$, an induced magnetic field $\vec{h} = (0, h, 0)$, an induced electric field $\vec{E} = (E_1, 0, E_3)$ and electric current density $\vec{J} = (J_1, 0, J_3)$ are developed in the medium Ω which satisfy the simplified linearized equations of electrodynamics of slowly moving continuous media having perfect electrical conductivity in absence of displacement current [30]:

$$\vec{J} = \vec{\nabla} \times \vec{h}, \tag{1}$$

$$\vec{\nabla} \times \vec{E} = -\mu_0 \dot{\vec{h}}, \tag{2}$$

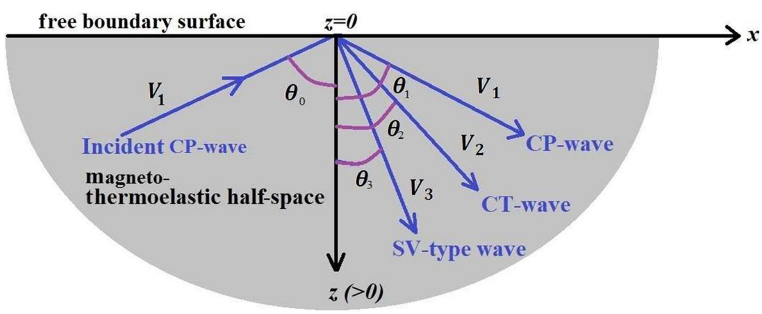


Figure 1. Schematic of the present problem: incident and reflected thermoelastic waves at the free surface $z = 0$.

$$\vec{E} = -\mu_0(\dot{\vec{u}} \times \vec{H}), \tag{3}$$

$$\nabla \cdot \vec{h} = 0, \tag{4}$$

where $\vec{H} = (0, H_0 + h, 0)$ is the total magnetic field and μ_0 is the magnetic permeability. The small effect of temperature gradient on $\vec{J}$ is ignored.

The above equations are supplemented by the following set of equations in the context of the MGL model [5]. Following [5], the basic equations of the MGL model in case of a homogeneous, isotropic, thermally and electrically conducting elastic medium (in absence of heat source) can be arranged in the following way (in general Cartesian coordinates system *Oxyz*):

- The equation of motion:

$$\left(1 + t_0\tau_1\frac{\partial}{\partial t}\right)[\mu u_{i,jj} + (\lambda+\mu)u_{j,ij}] - \gamma\left(1 + \tau_1\frac{\partial}{\partial t}\right)\Theta_{,i} + F_i = \rho\ddot{u}_i. \tag{5}$$

- The heat conduction equation in the present context

$$K_\Theta\Theta_{,ii} = \rho C_E\left(1 + \tau_0\frac{\partial}{\partial t}\right)\dot{\Theta} + \gamma T_0\left(1 + t_1\tau_0\frac{\partial}{\partial t}\right)\dot{e}_{kk}. \tag{6}$$

- The constitutive relation:

$$\sigma_{ij} = \left(1 + t_0\tau_1\frac{\partial}{\partial t}\right)(2\mu e_{ij} + \lambda e_{kk}\delta_{ij}) - \gamma\left(\Theta + \tau_1\dot{\Theta}\right)\delta_{ij}, \tag{7}$$

where u_i are the displacement components, $e_{ij} = (u_{i,j} + u_{j,i})/2$ are the components the elastic strain tensor, $e_{kk}(= u_{k,k})$ is the dilatation, σ_{ij} are the components of the elastic stress tensor, F_i are the components of the Lorentz's force vector, t is the time, λ, μ are Lamé's constants, $\gamma = (3\lambda + 2\mu)\alpha_T$ is the thermoelastic coupling parameter, α_T is the coefficient of linear thermal expansion, F_i are the components of body force vector, ρ is the mass density, C_E is the specific heat at constant strain, q_i are the components of the heat flux vector, η is the entropy, Θ is the thermodynamic temperature above the reference temperature T_0 such that $|\Theta/T_0| \ll 1$, τ_0, τ_1 are the thermal relaxation parameters such that $\tau_1 \geq \tau_0 \geq 0, K_\Theta$ is the thermal conductivity of the material of the medium and where t_0, t_1 are some constant parameters.

Equations (5)–(7) are the complete set of basic field equations in the context of the modified Green–Lindsay theory of thermoelasticity. Note that in the above equations, a comma followed by a suffix denotes *spatial derivative* and a *superposed dot* stands for time-differentiation. We shall consider Eqs. (1)–(7) as the basic governing equations for the present study. Equations (1)–(3) reduce to the particular set of equations for the MGL and the GL models when

- **MGL model:** $t_0 = t_1 = 1$.
- **GL model:** $t_0 = t_1 = 0$.

The Lorentz's force vector $\vec{F} = (F_1, F_2, F_3)$ is given by

$$\vec{F} = \mu_0(\vec{J} \times \vec{H}). \tag{8}$$

In the case of plane strain problem parallel to the xz–plane, all the field variables may be considered as functions of x , z and t only. Consequently, the displacement components and the temperature field may have the forms

$$u_1 = u(x,z,t),\;\; u_2 = v(x,z,t) = 0,\;\; u_3 = w(x,z,t),\;\; \Theta = \Theta(x,z,t).$$

Now, from Eqs. (1)–(4) and (8), we get

$$F_x = \mu_0 H_0^2 \frac{\partial e}{\partial x}, \quad F_y = 0, \quad F_z = \mu_0 H_0^2 \frac{\partial e}{\partial z}. \tag{9}$$

Hence, Eqs. (5)–(7) are simplified to

$$\left(1 + t_0\tau_1 \frac{\partial}{\partial t}\right)\left[\mu\nabla^2 u + (\lambda+\mu)\frac{\partial e}{\partial x}\right] - \gamma\left(1+\tau_1\frac{\partial}{\partial t}\right)\frac{\partial \Theta}{\partial x} + \mu_0 H_0^2 \frac{\partial e}{\partial x} = \rho\frac{\partial^2 u}{\partial t^2}, \tag{10}$$

$$\left(1 + t_0\tau_1 \frac{\partial}{\partial t}\right)\left[\mu\nabla^2 w + (\lambda+\mu)\frac{\partial e}{\partial z}\right] - \gamma\left(1+\tau_1\frac{\partial}{\partial t}\right)\frac{\partial \Theta}{\partial z} + \mu_0 H_0^2 \frac{\partial e}{\partial z} = \rho\frac{\partial^2 w}{\partial t^2}, \tag{11}$$

$$K_\Theta \nabla^2\Theta = \rho C_E\left(1+\tau_0\frac{\partial}{\partial t}\right)\frac{\partial\Theta}{\partial t} + \gamma T_0\left(1+t_1\tau_0\frac{\partial}{\partial t}\right)\frac{\partial e}{\partial t}, \tag{12}$$

$$\sigma_{xx} = \left(1+t_0\tau_1\frac{\partial}{\partial t}\right)\left(2\mu\frac{\partial u}{\partial x} + \lambda e\right) - \gamma\left(1+\tau_1\frac{\partial}{\partial t}\right)\Theta, \tag{13}$$

$$\sigma_{zz} = \left(1+t_0\tau_1\frac{\partial}{\partial t}\right)\left(2\mu\frac{\partial w}{\partial z} + \lambda e\right) - \gamma\left(1+\tau_1\frac{\partial}{\partial t}\right)\Theta, \tag{14}$$

$$\sigma_{xz} = \mu\left(1+t_0\tau_1\frac{\partial}{\partial t}\right)\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \tag{15}$$

where $e = (e_{xx} + e_{zz})$ and $\nabla^2 \equiv (\partial^2/\partial x^2 + \partial^2/\partial z^2)$.

In order to make the above equations dimensionless, let us introduce the following dimensionless parameters

$$(x', z') = C_L\eta(x,z), \;\; (u', w') = C_L\eta(u,w), \;\; (t', \tau_0', \tau_1') = C_L^2\eta(t, \tau_0, \tau_1) \;\; , \Theta' = \frac{\gamma\Theta}{\rho C_L^2}, \;\; \sigma_{ij}' = \frac{\sigma_{ij}}{\rho C_L^2},$$

where $C_L^2 = (\lambda + 2\mu)/\rho$ is the speed of the classical longitudinal (dilatational) wave and $\eta = \rho C_E/K_\Theta$ is the thermal viscosity.

With the help of the above variables, Eqs. (10)–(15) can be re-written as follows (we omit the primes for convenience):

$$\left(1 + t_0\tau_1 \frac{\partial}{\partial t}\right)\left[\beta^2\nabla^2 u + (1-\beta^2)\frac{\partial e}{\partial x}\right] - \left(1+\tau_1\frac{\partial}{\partial t}\right)\frac{\partial \Theta}{\partial x} + R_M \frac{\partial e}{\partial x} = \frac{\partial^2 u}{\partial t^2}, \tag{16}$$

$$\left(1 + t_0\tau_1 \frac{\partial}{\partial t}\right)\left[\beta^2\nabla^2 w + (1-\beta^2)\frac{\partial e}{\partial z}\right] - \left(1+\tau_1\frac{\partial}{\partial t}\right)\frac{\partial \Theta}{\partial z} + R_M \frac{\partial e}{\partial z} = \frac{\partial^2 w}{\partial t^2}, \tag{17}$$

$$\nabla^2\Theta = \left(1+\tau_0\frac{\partial}{\partial t}\right)\frac{\partial\Theta}{\partial t} + \varepsilon_\theta\left(1+t_1\tau_0\frac{\partial}{\partial t}\right)\frac{\partial e}{\partial t}, \tag{18}$$

$$\sigma_{xx} = \left(1+t_0\tau_1\frac{\partial}{\partial t}\right)\left[2\beta^2\frac{\partial u}{\partial x} + (1-2\beta^2)e\right] - \left(1+\tau_1\frac{\partial}{\partial t}\right)\Theta, \tag{19}$$

$$\sigma_{zz} = \left(1+t_0\tau_1\frac{\partial}{\partial t}\right)\left[2\beta^2\frac{\partial w}{\partial z} + (1-2\beta^2)e\right] - \left(1+\tau_1\frac{\partial}{\partial t}\right)\Theta, \tag{20}$$

$$\sigma_{xz} = \beta^2\left(1+t_0\tau_1\frac{\partial}{\partial t}\right)\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \tag{21}$$

where $\beta = \sqrt{\mu/(\lambda+2\mu)} = C_S/C_L$ is the ratio of the classical shear wave speed (C_S) to the classical longitudinal wave speed (C_L), $R_M = \mu_0 H_0^2/(\lambda+2\mu)$ is the magnetic pressure number, $\varepsilon_\Theta = \gamma^2 T_0/[\rho C_E(\lambda+2\mu)]$ is the dimensionless thermoelastic coupling constant.

Let us now introduce the displacement potentials ϕ and ψ through the Helmholtz vector representation as

$$u = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial z}, \quad w = \frac{\partial \phi}{\partial z} + \frac{\partial \psi}{\partial x}. \tag{22}$$

Substitution of Eq. (22) into Eqs. (16)–(18) yields

$$\left(1 + R_M + t_0\tau_1 \frac{\partial}{\partial t}\right)\nabla^2\phi - \frac{\partial^2 \phi}{\partial t^2} - \left(1 + \tau_1 \frac{\partial}{\partial t}\right)\Theta = 0, \tag{23}$$

$$\beta^2\left(1 + t_0\tau_1 \frac{\partial}{\partial t}\right)\nabla^2\psi - \frac{\partial^2 \psi}{\partial t^2} = 0, \tag{24}$$

$$\nabla^2\Theta = \left(1 + \tau_0 \frac{\partial}{\partial t}\right)\frac{\partial \Theta}{\partial t} + \varepsilon_\Theta\left(1 + t_1\tau_0 \frac{\partial}{\partial t}\right)\frac{\partial}{\partial t}(\nabla^2\phi). \tag{25}$$

Equations (23) and (25) show that the thermal field Θ is coupled with the displacement potential ϕ and results two sets of coupled-thermal-elastic waves, namely, a coupled-dilatational elastic wave (CP-wave) and a coupled-thermal wave (CT-wave). Equation (24) creates one independent modified shear-type wave (SV-type wave).

Dispersion equation and its solution

To seek the plane harmonic wave solutions of Eqs. (23)–(25) propagating in the positive direction of a unit vector **n** with speed c, we take the form of various potentials as [35,36]

$$\{\phi, \Theta, \psi\} = \{a_1, a_2, a_3\}\exp\{\imath k(\mathbf{n}\cdot\mathbf{r} - ct)\}, \tag{26}$$

where $a_1,\ a_2,\ a_3$ are the constants (possibly complex) representing the coefficients of the wave amplitudes, $\imath = \sqrt{-1},\ k$ is the assigned wavenumber and $\mathbf{r}$ is the position vector of a field point. The quantities k and c are connected with the angular frequency ω through the relation $\omega = kc$. Moreover, in view of dissipative character of the thermoelastic medium in question, we consider c as

$$c = \Re(c) + \imath\Im(c),$$

where $\Re(\cdot)$ and $\Im(\cdot)$ denote the real and imaginary parts of a complex number, respectively.

For the waves to be physically realistic, we should have

$$\Re(c) \geq 0 \quad \text{and} \quad \Im(c) \leq 0.$$

Here $\Re(c) \geq 0$ gives the wave speed, while $\Im(c) \leq 0$ describes the damping in time of the corresponding propagating wave. Also we note that,

- $\Im(c) = 0$ gives undamped wave in time;
- $\Im(c) < 0$ corresponds to a damped wave in time, decaying exponentially like $\exp[k\Im(c)t]$ to zero as time $t \to \infty$;
- $\Re(c) = 0$ together with $\Im(c) < 0$ generates a standing damped wave in time whose amplitude decays exponentially with time t.

Substituting the solutions (26) into the Eqs. (23)–(25), we get

$$\left[(1 + R_M - \imath\tau_1\omega t_0)k^2 - \omega^2\right]a_1 + (1 - \imath\tau_1\omega)a_2 = 0, \tag{27}$$

$$\left[\beta^2(1 - \imath\tau_1\omega t_0)k^2 - \omega^2\right]a_3 = 0, \tag{28}$$

$$\imath\varepsilon_\Theta\omega(1-\imath\tau_0\omega t_1)k^2a_1+[k^2-\imath\omega(1-\imath\tau_0\omega)]a_2=0. \tag{29}$$

The condition for the existence of the non-trivial solutions for a_1 and a_2 of the system of Eqs. (27) and (29) yield the following dispersion equation

$$\left[(1+R_M-\imath\tau_1\omega t_0)k^2-\omega^2\right]\left[k^2-\imath\omega(1-\imath\tau_0\omega)\right]-\imath\varepsilon_\Theta\omega(1-\imath\tau_0\omega t_1)(1-\imath\tau_1\omega)k^2=0. \tag{30}$$

For MGL model ($t_0=t_1=1$), Eq. (30) reduces to

$$(1+R_M-\imath\tau_1\omega)k^4-\left[\imath\omega(1+\varepsilon_\Theta+R_M)(1-\imath\tau_0\omega)(1-\imath\tau_1\omega)+\omega^2\right]k^2+\imath\omega^3(1-\imath\tau_0\omega)=0, \tag{31}$$

which is the general dispersion relation for the coupled dilatational thermal-elastic waves in the context of MGL model of generalized thermoelasticity.

In case of GL model ($t_0=t_1=0$), we obtain from Eq. (30) that

$$(1+R_M)k^4-\left[\imath\omega(1+R_M)(1-\imath\tau_0\omega)+\imath\varepsilon_\Theta\omega(1-\imath\tau_1\omega)+\omega^2\right]k^2+\imath\omega^3(1-\imath\tau_0\omega)=0, \tag{32}$$

Equation (32) is the dispersion equation of the coupled dilatational thermal-elastic waves under the GL model as reported by Agarwal [14] in presence of magnetic field effect ($R_M=0$). Again, setting $\tau_1=\tau_0=\tau$ and $R_M=0$, Eq. (32) agrees with the corresponding result of Nayef and Nemat-Nasser [12].

Moreover, if we substitute $\tau_0=\tau_1=0$ and neglect the magnetic field effect from the medium, then the dispersion relation (30) reduces to

$$k^4-k^2\left[\omega^2+\imath\omega(1+\varepsilon_\theta)\right]+\imath\omega^3=0, \tag{33}$$

which is the dispersion relation for the coupled thermoelastic plane wave propagation under the CTE theory as reported by Chadwick and Sneddon [9].

So, Eq. (30) is the more general dispersion relation for the coupled magneto-thermal-elastic waves propagation in the frame of MGL, GL and CT models. For a given ω, Eq. (30) gives us four roots of the form $\pm k_1$ and $\pm k_2$, for k. Of these four roots, only two roots yield positive values for $\Re(k)$ with $\Im(k_{1,2})\geq 0$. Hence, there are two distinct traveling coupled-dilatational elastic-thermal waves of wavenumber $k_{1,2}$, namely, a CP-wave and a CT-wave. Both of the waves are influenced by the strain rate present in the MGL model. The magnetic field also affects these waves. The phase speeds of the CP- and CT-waves are given by $V_{1,2}=\omega/\Re(k_{1,2})$. Since the attenuation coefficients, $\Im(k_{1,2})$ and the phase speeds, $V_{1,2}$ are nonlinear functions of ω, these waves suffer attenuation as well as dispersion due to the magneto-thermoelastic character of the medium considered. The magnetic field and thermoelastic material properties of the medium and the strain rate term of MGL model influence the dispersion and the attenuation of these waves. Besides, since the wavenumbers of both the waves are complex, so they are inhomogeneous waves. The CP- and CT-waves are coupled-dilatational elastic-thermal waves and the coupling is measured by the following ratio of the amplitude:

$$\left(\frac{a_2}{a_1}\right)_j=\frac{\left[\omega^2-(1+R_M-\imath\tau_1\omega t_0)k^2\right]}{(1-\imath\tau_1\omega)}=\frac{\varepsilon_\Theta\omega(1-\imath\tau_1\omega t_1)k_j^2}{\left[\omega(1-\imath\tau_0\omega)+\imath k_j^2\right]}=\zeta_j \quad (j=1,2). \tag{34}$$

A look at the Eq. (28) reveals that there exist one SV-type wave of wavenumber $k_3=\omega/[\beta\sqrt{1-\imath\tau_1\omega t_0}]$ whose phase speed V_3 is given by

$$V_3=\frac{\omega}{\Re(k_3)}. \tag{35}$$

The expression in (35) clearly shows that the SV-type wave is affected by the presence of the strain rate in MGL model but it is not affected by the thermal wave. It is also interesting to note that this wave is dispersive in nature and exhibits attenuation in the case of MGL model in contrast to the GL and the CTE models.

Perturbation solution of dispersive waves

The perturbation method has been widely used (Nayfeh and Nemat-Nasser [12], Agarwal [14], Roy Choudhuri [15, 18], Sharma et al. [37]) to study the wave propagation problems in classical (coupled) and non-classical (generalized) thermoelastic continua. To explore and delineate the CP- and CT-waves solutions, we seek solutions of the dispersion equation for small thermoelastic coupling parameter ε_Θ $(\ll 1)$. Equation (30) for $\varepsilon_\Theta = 0$ admits the solutions

$$J_1^2 = \frac{\omega^2}{(1 + R_M - \imath\tau_1\omega t_0)}, \quad J_2^2 = \imath\omega(1 - \imath\tau_0\omega). \tag{36}$$

Thus for the present problem, we conclude that while J_1 corresponds to a CP-wave, J_2 corresponds to a CT-wave, both modified by the presence of the magnetic field effect and the strain rate of MGL.

For most of the thermoelastic materials, the parameter ε_Θ is very small and therefore, we develop series expansions in terms of ε_Θ for the roots of the Eq. (40) in order to explore the effect of various parameters of interest on these waves. Thus, for $\varepsilon_\Theta \ll 1$, we now set

$$k^2 = k_1^2 = J_1^2 + \delta_u\varepsilon_\theta + O(\varepsilon_\Theta^2), \tag{37}$$

$$k^2 = k_2^2 = J_2^2 + \delta_\theta\varepsilon_\theta + O(\varepsilon_\Theta^2). \tag{38}$$

Substitute (37) and (38) into (30) and equate the coefficients of like powers of ε_Θ, we obtain

$$\delta_u = \frac{\imath\omega(1 - \imath\tau_0\omega t_1)(1 - \imath\tau_1\omega)J_1^2}{(1 + R_M - \imath\tau_1\omega t_0)(J_1^2 - J_2^2)}, \quad \delta_\theta = \frac{\imath\omega(1 - \imath\tau_0\omega t_1)(1 - \imath\tau_1\omega)J_2^2}{(1 + R_M - \imath\tau_1\omega t_0)(J_2^2 - J_1^2)}.$$

Following Roy Choudhuri [15] and Sharma et al. [37], we may call k_1 and k_2 as the wavenumber of CP- and CT-waves, respectively. For GL model without magnetic field effect $(t_0 = t_1 = 0,\ R_M = 0)$, the solutions k_1 and k_2 agree with those obtained by Roy Choudhuri [15] for non-rotating media. Again, setting $\tau_1 = \tau_0 = \tau,\ R_M = 0$ and $t_0 = t_1 = 0,$ we note that k_1 and k_2 are exactly the same expressions as reported by Nayfeh and Nemat-Nasser [12]. In case of the CTE model $(\tau_0 = \tau_1 = 0, R_M = 0)$, the expressions k_1 and k_2 are in complete agreement with those obtained by Chadwick and Sneddon [9].

Reflection phenomena of magneto-thermoelastic waves

Let a train of CP-wave having amplitude A_0 and phase speed V_1 is made incident on the free surface $z=0$ making an angle θ_0 with the normal to $z=0$ as shown in Figure 1. Assuming that the radiation in vacuum is neglected, when it impinges the boundary $z=0$, three reflection waves in the medium are created. Suppose the reflected CP-, CT- and SV-type waves make angles $\theta_1,\ \theta_2$ and θ_3, respectively with the positive z–axis. Then the complete structure of the wave field consisting of the incident and reflected waves in the medium Ω may be written as

$$\phi = A_0 \exp\{\imath k_1(x\sin\theta_0 - z\cos\theta_0) - \imath\omega t\} + \sum_{j=1}^{2} A_j \exp\{\imath k_j(x\sin\theta_j + z\cos\theta_j) - \imath\omega t\}, \tag{39}$$

$$\Theta = \zeta_1 A_0 \exp\{\imath k_1(x\sin\theta_0 - z\cos\theta_0) - \imath\omega t\} + \sum_{j=1}^{2} \zeta_j A_j \exp\{\imath k_j(x\sin\theta_j + z\cos\theta_j) - \imath\omega t\}, \tag{40}$$

$$\psi = B_1 \exp\{\imath k_3(x\sin\theta_3 + z\cos\theta_3) - \imath\omega t\}. \tag{41}$$

where A_1, A_2 and B_1 represent the coefficients of amplitudes of the reflected CP-, CT- and SV-type waves respectively, and ζ_j, $j = 1, 2$ are listed in Eq. (34). The reflection coefficients are defined as the ratios of the amplitudes of the reflected to the incident wave and are determined by the well-defined boundary conditions on the surface $z = 0$.

Boundary conditions: Stress-free thermally insulated surface

We consider the surface $z = 0$ as stress-free and thermally insulated. These conditions may be mathematically expressed as follows:

Mechanical boundary conation

$$\sigma_{zz} + \bar{\tau}_{zz} = \sigma_{xz} + \bar{\tau}_{xz} = 0, \quad \text{at} \quad z = 0, \tag{42}$$

where Maxwell's electro-magneto stress tensor $\bar{\tau}_{ij}$ is given by

$$\bar{\tau}_{ij} = \mu_0[H_i h_j + H_j h_i - (\vec{H}\cdot\vec{h})\delta_{ij}], \quad i,j = 1,2,3. \tag{43}$$

Thermal boundary conation

$$\frac{\partial\Theta}{\partial z} = 0, \quad \text{at} \quad z = 0. \tag{44}$$

In terms of displacement potential functions, (42) can be written in non-dimensional forms as

$$\left(1 + R_M + \tau_1 t_0\frac{\partial}{\partial t}\right)\left(\frac{\partial^2\phi}{\partial z^2} + \frac{\partial^2\phi}{\partial x^2}\right) + 2\beta^2\left(1 + \tau_1 t_0\frac{\partial}{\partial t}\right)\left(\frac{\partial^2\psi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial x^2}\right) - \left(1 + \tau_1\frac{\partial}{\partial t}\right)\Theta = 0, \tag{45}$$

$$\left(2\frac{\partial^2\phi}{\partial x\partial z} + \frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial z^2}\right) = 0, \quad \text{at} \quad z = 0. \tag{46}$$

In order to satisfy the above boundary conditions at $z = 0$, we apply the Snell's law which leads to

$$k_1\sin\theta_0 = k_1\sin\theta_1 = k_2\sin\theta_2 = k_3\sin\theta_3, \tag{47}$$

or in the form

$$\theta_0 = \theta_1 \quad \text{and} \quad \frac{\sin\theta_0}{V_1} = \frac{\sin\theta_2}{V_2} = \frac{\sin\theta_3}{V_3}, \tag{48}$$

which is often refereed as extended Snell's law.

Substituting from Eqs. (39)–(41) into (44)–(46) and using the relation (47) or (48), the following system of equations for the reflection coefficients $R_{CP} = A_1/A_0$, $R_{CT} = A_2/A_0$, $R_{SV} = B_1/A_0$ is obtained:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & 0 \end{bmatrix}\begin{bmatrix} R_{CP} \\ R_{CT} \\ R_{SV} \end{bmatrix} = \begin{bmatrix} -a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}, \tag{49}$$

where

$$a_{11} = \omega^2 - 2\beta^2(1 - \imath\tau_1\omega t_0)k_1^2 \sin^2\theta_0, \quad a_{12} = \omega^2 - 2\beta^2(1 - \imath\tau_1\omega t_0)k_2^2 \sin^2\theta_2, \quad a_{13} = \omega^2 \sin 2\theta_3,$$
$$a_{21} = k_1^2 \sin 2\theta_0, \quad a_{22} = k_2^2 \sin 2\theta_2, \quad a_{23} = -k_3^2 \cos 2\theta_3,$$
$$a_{31} = \zeta_1 k_1 \cos\theta_1, \quad a_{32} = \zeta_2 k_2 \cos\theta_2.$$

Solving (49), we get the reflection coefficients in explicit forms. It is quite interesting to note that these reflection coefficients are dependent on the angle of incidence (θ_0), strain rate of MGL, magnetic field (H_0) and the material properties of the medium. It can also be noted that for uncoupled thermoelasticity ($\varepsilon_\Theta = 0$), $\zeta_j = 0$ $(j = 1, 2)$, and hence there is no reflected CT-wave. So, in this case $R_{CT} = 0$ at all angle of incidence θ_0.

Energy partition

In order to physically justify the analytic expressions of the reflection coefficients in the present problem, we must verify the energy balance law at the boundary surface $z = 0$. Let us consider the energy partition between various reflected waves at a surface element of the unit area. Following [36], the rate of energy transmission, P per unit area at a free surface of a thermoelastic solid is given by

$$P = (\sigma_{zz} + \bar{\tau}_{zz})\frac{\partial w}{\partial t} + (\sigma_{xz} + \bar{\tau}_{xz})\frac{\partial u}{\partial t}. \tag{50}$$

Note that here, the contribution of thermal energy as well as the interaction energy is negligibly so small compared to the other energy terms. Also even if these energies are accounted in Eq. (50), these do not change the results qualitatively. However, some physical situations may arise where the contribution of thermal energy as well as the interaction energy is comparable to the other energy and in that case it is essential to include these energies in Eq. (50) (cf. Li et al. [38]).

Let $\langle P_0 \rangle$ denotes the average energy carried along with incident CP-wave, $\langle P_j \rangle$ $(j = 1, 2)$, respectively denote the average energy carried along the reflected CP- and CT-waves and $\langle P_3 \rangle$ denotes the average energy carried along reflected SV-type wave.

We define energy ratio E_j $(i = 1, 2, 3)$ corresponding to the j–th reflected wave at $z = 0$ as the ratio of energy carried along j–th reflected wave to the energy carried along the incident CP-wave:

$$E_j = \frac{\langle P_j \rangle}{\langle P_0 \rangle}. \tag{51}$$

Thus, for an incident CP-wave having phase speed V_1, the analytical expressions of the energy ratios E_{CP}, E_{CT} and E_{SV} of the reflected CP-, CT- and SV-type waves, respectively are obtained by using Eqs. (20)–(22), (39)–(41), (43), (50) and (55) as follows:

$$E_{CP} = -R_{CP}^2, \quad E_{CT} = -\frac{\tan\theta_0}{\tan\theta_2}R_{CT}^2, \quad E_{SV} = -\frac{\tan\theta_0}{\tan\theta_3}R_{SV}^2, \tag{52}$$

where $0^\circ < \theta_0 < 90^\circ$. Like the amplitude ratios, the energy ratios also depend on θ_0, material properties of the thermoelastic medium Ω, and the amplitude ratios. Since, surface waves are not involved in the energy conservation principle, so the conservation of energy at the surface $z = 0$ may be stated as:

$$E_{sum} = |E_{CP} + E_{CT} + E_{SV}| \approx 1. \tag{53}$$

Numerical example and discussions

In this section, we perform some numerical calculations in order to illustrate the analytical results. For this purpose, we choose copper like material whose physical data are [33]:

$$\lambda = 7.76 \times 10^{10}\ Nm^{-2},\quad \mu = 3.86 \times 10^{10}\ Nm^{-2},\quad T_0 = 293\ K,\quad \rho = 8954\ kgm^{-3},$$
$$C_E = 383.1\ Jkg^{-1}K^{-1},\quad K_T = 386\ Wm^{-1}K^{-1},\quad \alpha_T = 1.78 \times 10^{-5}\ K^{-1},\quad \varepsilon_\Theta = 0.0168.$$

Following Othman and Song [21,22], we select the other parameters as: $R_M = 0.5,$ $\tau_0 = 0.05,$ $\tau_1 = 0.075.$

The effect of the magnetic pressure number R_M on the variations of the absolute values of the reflection coefficients and the energy ratios with the angle of incidence θ_0 in the range $0^\circ \leq \theta_0 \leq 90^\circ$ has been depicted through Figure 2a–f. In calculation, three different values of the magnetic pressure number, that is, $R_M = 0.0,\ 0.5,\ 1.0,$ are set while the Poison's ratio is prescribed as $\sigma = 0.33.$ Figure 2a shows that R_M produces an increasing effect on the reflection coefficient $|R_{CP}|$ of the reflected CP-wave. On the contrary, we observe in Figure 2b that the magnetic pressure number R_M has a decreasing effect on the reflection coefficients $|R_{CT}|$ and $|R_{SV}|$ of the reflected CT-wave and the SV-type wave, respectively. It is also noticed that the modulus of R_{CP} is the highest and that of $|R_{CT}|$ is the least. The maxima of $|R_{CP}|$ are occurred at $\theta_0 = 0^\circ$ and 90° for each R_M. Figure 2b reveals that, all the curves converge to zero at the grazing incidence $(\theta_0 = 90^\circ)$ in all the cases. It is evident in Figure 2c that $|R_{CT}|$ attains its minima at $\theta_0 = 0^\circ$ (normal incidence) and 90° 9grazing incidence0 for each R_M whereas this quantity attains its maximum near $\theta_0 = 45^\circ$ in absence of the magnetic field effect $(R_M = 0.0)$. It is observed in Figure 2d–f that the pattern of each of the energy ratio $(|E_{CP}|,\ |E_{CT}|,\ |E_{SV}|)$ is qualitatively similar to the pattern of the corresponding reflection coefficient. It is quite appealing as the energy ratios are proportional to the square of the corresponding reflection coefficients at each θ_0. It is also interesting to note that the energy ratio $|E_{CT}|$ of the reflected CT-wave is very small as compared to the absolute values of the reflection coefficients of the reflected CP- and SV-type waves. Thus the energy carried along the reflected CT-wave is least which in turn means that the maximum amount of the incident energy is carried along the reflected CP- and the SV-type waves.

The influence of the Poison's ratio σ upon the variations of the moduli of reflection coefficients as well as the energy ratios are also interesting [36] and have been presented in Figure 3a–f. In calculation, three different values of the Poison's ratio, that is, $\sigma = 0.30,\ 0.33,\ 0.36,$ are set while the magnetic pressure number parameter is prescribed as $R_M = 0.5$. It is evident from these figures that the Poison's ratio has an increasing effect on $|R_{CP}|$ and $|R_{CT}|$ while it shows a decreasing effect on $|R_{SV}|$. The reflection coefficients of the CP-wave and the SV-type wave are most sensitive to σ whereas the reflection coefficient of the CT-wave is most insensitive to σ.

The key point noticed from the Figures 2a–c and 3a–c is that the magnetic pressure number and the Poison's ratio have a larger effect on $|R_{CP}|$ and $|R_{SV}|$ as compared to $|R_{CT}|$. This is most probably due to the following mechanism: as shown in the Eqs. (16) and (17), the magnetic pressure number and the Poison's ratio are directly presented into the equation of motion instead of the heat conduction equation to characterize the effects of these parameters on the reflection coefficients of the various reflected waves, which in turn leads to the consequence that these parameters barely influences the reflection coefficient, $|R_{CT}|$ of the reflected CT-wave.

Figure 4a–c are plotted to examine the influence of the thermoelastic coupling parameter ε_Θ on the variations of moduli of the reflection coefficients against the angle of incidence θ_0. In calculation, three different values of the thermoelastic coupling parameter, that is, $\varepsilon_\Theta = 0.0,\ 0.0168,\ 0.0336,$ are set while the magnetic pressure number and the Poison's ratio are prescribed as $R_M = 0.5$ and $\sigma = 0.33,$ respectively. The case $\varepsilon_\Theta = 0.0$ corresponds to the uncoupled thermoelasticity theory. Similar to the effect of Poison's ratio, the thermoelastic coupling parameter ε_Θ has also had an increasing effect on $|R_{CP}|$ and $|R_{CT}|$ while it shows a decreasing effect on $|R_{SV}|$. In Figure 2b, it is depicted that for $\varepsilon_\Theta = 0.0,$ the quantity $|R_{CT}|$ vanishes identically in the

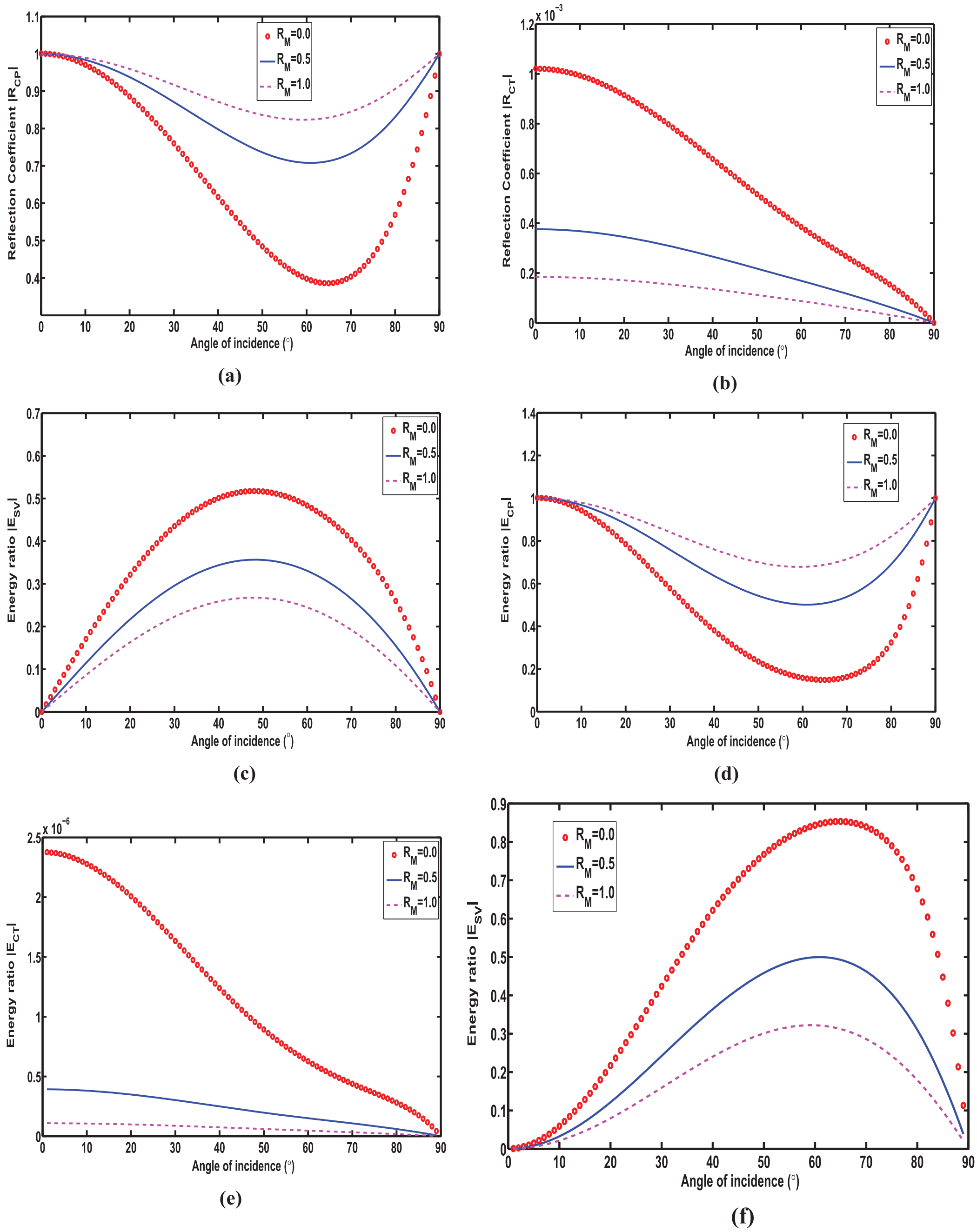


Figure 2. (a–f) Effect of the magnetic pressure number R_M on the variations of the reflection coefficients and the energy ratios for MGL model.

whole range of θ_0 which in turn means that there will be no reflection of the CT-wave in this case. This is the verification of a result pointed out in the text.

The key point noticed from Figure 4a–c is that the thermoelastic coupling parameter has a larger effect on $|R_{CT}|$ as compared to $|R_{CP}|$ and $|R_{SV}|$. This is most probably due to the following mechanism: as shown in the Eq. (18) or (25), the thermoelastic coupling parameter is directly introduced into the heat conduction equation instead of the constitutive relations to characterize

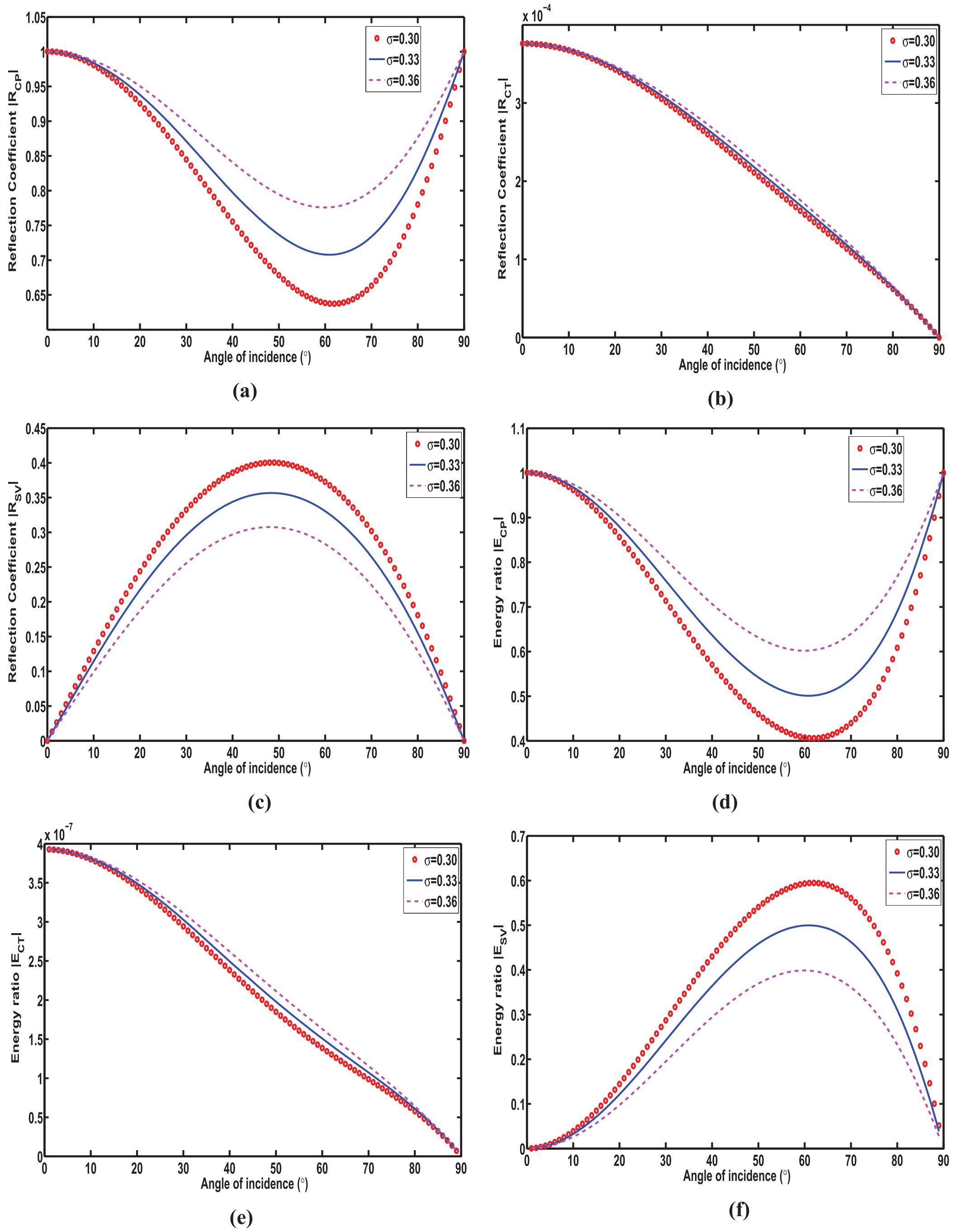
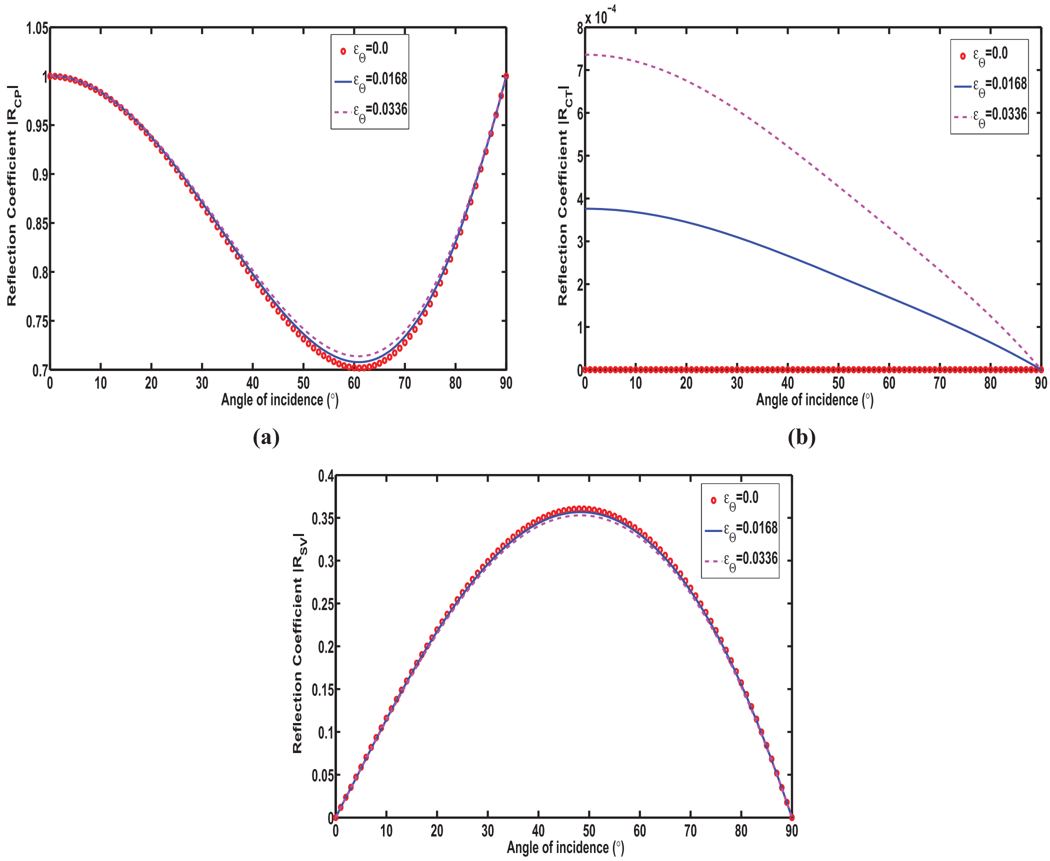


Figure 3. (a–f) Effect of the Poison's ratio σ on the variations of the reflection coefficients and the energy ratios for MGL model.

the effects of the thermoelastic coupling parameter on the reflection coefficients of the various reflected waves.

Figure 5a is drawn to compare the reflection coefficients $|R_{CP}|, |R_{CT}|$ and $|R_{SV}|$ with respect to θ_0. It is shown that the reflection coefficient $|R_{CP}|$ is remaining greater when compared to those



(c)

Figure 4. (a–c) Effect of the ε_Θ on the variations of the reflection coefficients for MGL model.

of $|R_{CT}|$ and $|R_{SV}|$. In order to validate the energy balance law at the thermally insulated stress-free surface $z = 0$, the absolute values of the energy ratios E_{CP}, E_{CT}, E_{SV} carried along the reflected CP-wave, CT-wave, SV-type wave, respectively and their sum E_{sum} (energy conservation index) have been calculated for different θ_0 and presented graphically in Figure 5b when $R_M = 0.5,\ \sigma = 0.33$. The energy ratio $|R_{CT}|$ is plotted after mounting up its original value by 10^6. It is observed that the energy conservation index, keeps unit value nearly at each θ_0 which verifies that the energy balance law at the free surface $z = 0$ is satisfied. However, Figure 5c reveals a smaller deviation from the unity of the energy conservation index in the presence $(R_M = 0.5,\ 1.0)$ as well as the absence of the magnetic field $(R_M = 0.0)$. This is attributed to the loss of numerical precision. The approximate satisfaction of the energy conservation law validates the present numerical results to a large extent. It is also interesting to evident that in presence of the magnetic field, more accuracy in the energy conservation index can be expected.

Figure 6a–f is drawn to make a comparison of the reflection coefficients and the corresponding energy ratios of the reflected waves obtained in the case of MGL and GL model with respect to the angle of incidence θ_0 when $R_M = 0.5$. We have shown that the reflection coefficient $|R_{CP}|$ is remaining greater for GL model when compared to that for MGL model while reverse natures have been observed for $|R_{CT}|$ and $|R_{SV}|$. As the energy ratios are proportional to the square of the corresponding reflection coefficients at each θ_0, it is observed in Figure 6d–f that the pattern of each of the energy ratio is qualitatively similar to the pattern of the corresponding reflection coefficient for both the model MGL as well as GL.

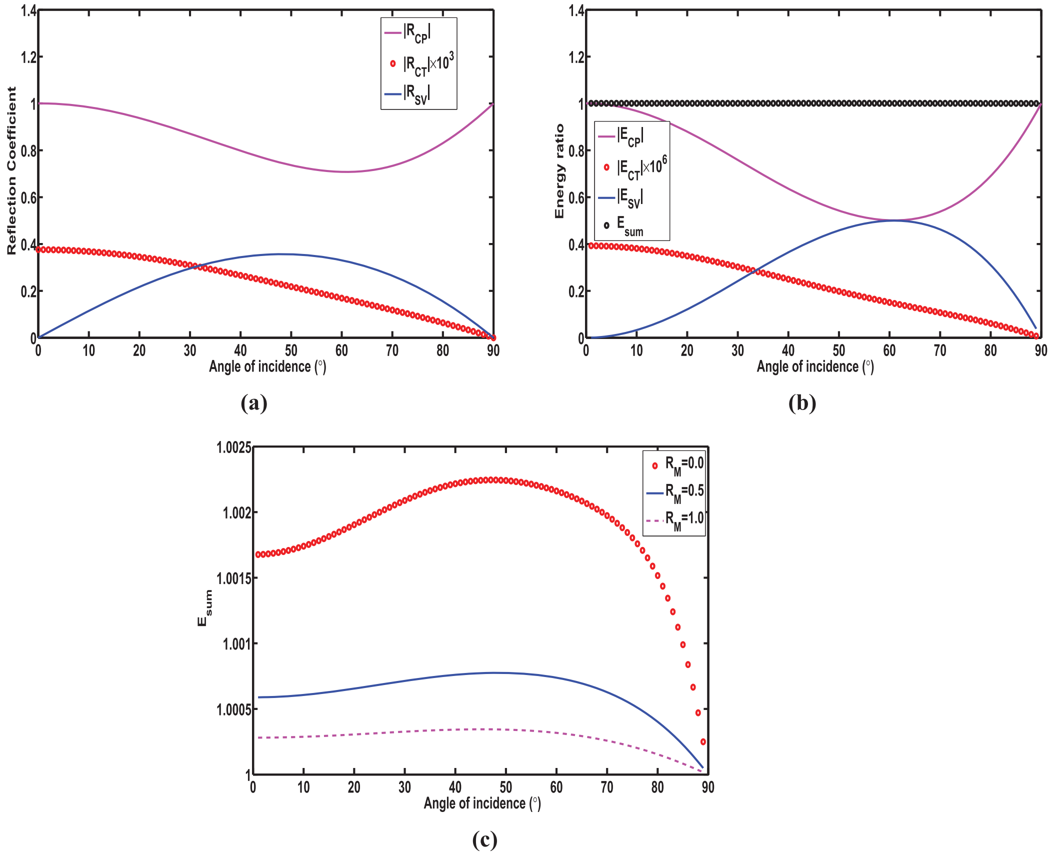


Figure 5. (a–c) Comparison of the (a) reflection coefficients and (b) energy ratios with respect to θ_0 for MGL model. (c) Variation of the energy balance index, E_{sum} in absence and presence of the magnetic field in MGL model.

Conclusions

In a generalized thermoelastic medium under the MGL theory, there are a total of three kinds of propagating waves. The thermoelastic coupling generates two sets of coupled longitudinal waves, namely a CP-wave and a CT-wave. There is also one independent SV-type wave. The reflection of thermoelastic waves is also studied for the incident CP-wave at a stress-free insulated surface. A numerical example is provided and the following conclusions can be drawn based on these numerical results.

1. The thermal field affects only the coupled longitudinal waves. The coupling between the displacement and the temperature fields makes the coupled longitudinal waves not only dispersive but also attenuated. Similarly, the introduction of the MGL model makes the SV-type wave not only dispersive but also attenuated in contrast to the other generalized thermoelastic models (GL, LS etc.).
2. The reflection coefficients and corresponding energy ratios are functions of the magnetic field, angle of incidence and thermoelastic parameter of the medium.
3. Numerical results show that the reflection coefficients and the respective energy ratios of the reflected CP- and SV-type waves are significantly affected by the Poison's ratio. On the contrary, ε_Θ affects significantly the reflection coefficients of the reflected CT-wave only.

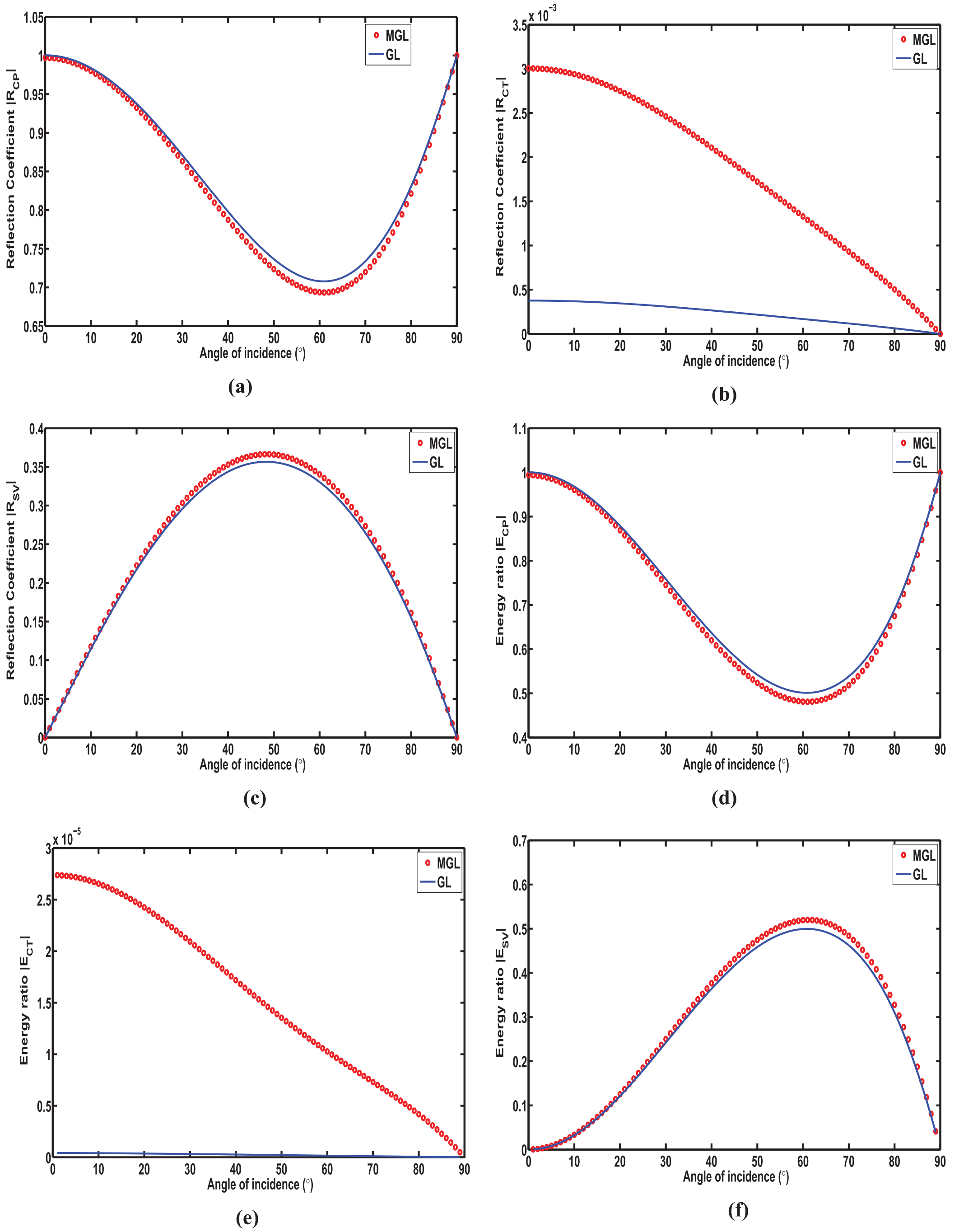


Figure 6. (a–f) Comparison of the reflection coefficients and the corresponding energy ratios of the reflected waves in the case of MGL and GL model when $R_M = 0.5$.

4. It is observed that the maximum amount of the incident energy is carried along the reflected CP-wave and the SV-type wave.
5. The numerical results depict that the energy conservation index keeps unity at each angle of incidence, thus proving the principle of conservation of energy. It is found that there is no dissipation of energy at the plane boundary surface $z = 0$ during the reflection of the waves.

6. To the best knowledge of the authors, there is no published work on reflection of magneto-thermoelastic waves on the basis of the new MGL theory of generalized thermoelasticity to date. The present work is very much expected to be useful for investigating various wave propagation problems, both theoretically and in observation of wave propagation. In particular, the present work is of geophysical interest for investigations of earthquakes and similar phenomena in seismology and engineering. Finally, the authors believe that the present theoretical and numerical results may provide interesting and significant information for experimental scientists, researchers, and seismologists working on this type of problems.

Disclosure statement

The authors declare that they have no Conflict of interest.

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